

## Covariant symplectic structure of the complex Monge-Ampère equation

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### abstract :

The complex Monge-Ampère equation is invariant under arbitrary holomorphic changes of the independent variables with unit Jacobian. We present its variational formulation where the action remains invariant under this infinite group. The new Lagrangian enables us to obtain the first symplectic 2-form for the complex Monge-Ampère equation in the framework of the covariant Witten-Zuckerman approach to symplectic structure. We base our considerations on a reformulation of the Witten-Zuckerman theory in terms of holomorphic differential forms. The first closed and conserved Witten-Zuckerman symplectic 2-form for the complex Monge-Ampère equation is obtained in arbitrary dimension and for all cases elliptic, hyperbolic and homogeneous.

The connection of the complex Monge-Ampère equation with Ricci-flat Kähler geometry suggests the use of the Hilbert action principle as an alternative variational formulation. However, we point out that Hilbert's Lagrangian is a divergence for Kähler metrics and serves as a topological invariant rather than yielding the Euclideanized Einstein field equations. Nevertheless, since the Witten-Zuckerman theory employs only the boundary terms in the first variation of the action, Hilbert's Lagrangian can be used to obtain the second Witten-Zuckerman symplectic 2-form. This symplectic 2-form vanishes on shell, thus defining a Lagrangian submanifold. In its derivation the connection of the second symplectic 2-form with the complex Monge-Ampère equation is indirect but we show that it satisfies all the properties required of a symplectic 2-form for the complex elliptic, or hyperbolic Monge-Ampère equation when the dimension of the complex manifold is 3 or higher.

The complex Monge-Ampère equation admits covariant bisymplectic structure for complex dimension 3, or higher. However, in the physically interesting case of  $n = 2$  we have only one symplectic 2-form.

The extension of these results to the case of complex Monge-Ampère-Liouville equation is also presented.

# 1 Introduction

The complex Monge-Ampère equation plays a central role in physics and mathematics. On a complex manifold  $\mathcal{M}$  of dimension  $n$  it is given by

$$\frac{1}{n!}(\partial\bar{\partial}u)^n = \kappa * 1, \quad \kappa = \pm 1, 0 \quad (1)$$

where  $\partial$  is the holomorphic exterior derivative,  $\bar{\partial}$  is its anti-holomorphic counter-part, the wedge product is understood and  $*1$  denotes the volume element. It will be referred to as  $CMA_n$ . Perhaps the more familiar form of this equation employs the Monge-Ampère determinant

$$\mu \equiv \det u_{i\bar{k}} = \frac{1}{n!} * (\partial\bar{\partial}u)^n = \kappa \quad (2)$$

where

$$u_{i\bar{k}} \equiv \frac{\partial^2 u}{\partial \zeta^i \partial \bar{\zeta}^k}, \quad (3)$$

and  $*$  is the Hodge star duality operator. Depending on  $\kappa = \pm 1$   $CMA_n$  is elliptic, or hyperbolic respectively and it is called homogeneous for  $\kappa = 0$ . Chern, Levine and Nirenberg [1] have pointed out that the homogeneous  $CMA_n$  is the fundamental equation in the theory of functions of many complex variables. The Laplace equation itself is  $CMA_1$ . In differential geometry elliptic  $CMA_n$  is the equation governing the Kähler potential for metrics with Euclidean signature and vanishing first Chern class [2]. This was noted by Calabi [3] and Yau [4] has given an existence proof. The Riemann curvature  $(1,1)$ -form is then self-dual which by the first Bianchi identity implies Ricci-flatness. These metrics are hyper-Kähler [5]. Furthermore, since Ricci-flat metrics satisfy the vacuum Einstein field equation with Euclidean signature,  $CMA_2$  is of vital interest for general relativistic instantons. The construction of the Riemannian metric for the gravitational instanton  $K3$ , also familiar as Kummer's surface [6], still remains as the outstanding unsolved problem of elliptic  $CMA_2$ .

Recently we have found [7] that real Monge-Ampère equations admit rich multi-symplectic structure. In this letter we shall present the symplectic structure of  $CMA_n$  which, in spite of its radical differences with the real case, also admits interesting multi-symplectic structure.  $CMA_n$  is invariant under arbitrary holomorphic changes of the independent variables. In the elliptic and hyperbolic cases we must add the proviso that the Jacobian must be unity [8]. Clearly, the symplectic structure of  $CMA_n$  must be covariant

under this infinite group. The usual approach to symplectic structure starts with a choice of time parameter, thus breaking covariance from the outset. It is therefore not suitable for a discussion of  $CMA_n$ . Happily, Witten [9] and Zuckerman [10] have shown that there exists a covariant, closed and conserved 2-form vector density, the time component of which gives the familiar symplectic 2-form. We shall use the Witten-Zuckerman approach to symplectic structure, reformulating it in order to express everything in terms of holomorphic and anti-holomorphic differential forms.

Finally, we note that the usual discussion of the symplectic structure of  $CMA_n$ , cf [11], is based on the Kähler  $(1, 1)$ -form

$$\omega_K = \frac{1}{2i} \partial \bar{\partial} u \quad (4)$$

but this is not the relevant object that emerges from our examination of symplectic structure. Our approach to symplectic structure will be in the framework of dynamical systems with infinitely many degrees of freedom [12]. We shall start with action principles underlying  $CMA_n$  and from the Lagrangian derive the corresponding Witten-Zuckerman symplectic 2-form. The results we shall present for  $CMA_n$  are quite different from the usual considerations using the Kähler  $(1, 1)$ -form.

## 2 First symplectic structure of $CMA_n$

Everything that is of interest is derivable from a variational principle [13]  $\delta I = 0$ ,

$$I = \int L \quad (5)$$

where  $L$  is the Lagrangian volume form. For  $CMA_n$  the Lagrangian is an  $(n, n)$ -form

$$L = \frac{1}{(n+1)!} \partial u \bar{\partial} u (\partial \bar{\partial} u)^{n-1} + \kappa u^* 1 \quad (6)$$

and it can be verified directly that first variation of the action (5) yields  $CMA_n$ . This Lagrangian has not been considered before and the easiest way of remembering it is through its determinantal structure

$${}^*L = \det \begin{vmatrix} 0 & u_i \\ u_{\bar{k}} & u_{i\bar{k}} \end{vmatrix} + \kappa u \quad (7)$$

which should be compared to the Monge-Ampère determinant (2). The Lagrangian (6) is also a linear combination of the zeroth and second order differential invariants in the prolongation structure and group foliation of  $CMA_n$  [14].

We shall use a reformulation of the Witten-Zuckerman theory in terms of holomorphic and anti-holomorphic differential forms to obtain the symplectic structure of  $CMA_n$ . The first variation of the Lagrangian (6) yields

$$\delta L = \partial\alpha + (-1)^n \bar{\partial}\bar{\alpha} + (\kappa - \mu) \delta u * 1 \quad (8)$$

where  $\alpha$  is given by

$$\alpha = \frac{1}{2n!} \delta u \bar{\partial} u (\partial \bar{\partial} u)^{n-1} + \frac{n-1}{2(n+1)!} \bar{\partial} \delta u \partial u \bar{\partial} u (\partial \bar{\partial} u)^{n-2} \quad (9)$$

which is an  $(n-1, n)$ -form on  $\Lambda^{n-1, n}(\mathcal{M})$  as well as a 1-form on  $\Lambda^1(\mathcal{F}(\mathcal{M}))$ , the space of functions over  $\mathcal{M}$ . We note that in Witten-Zuckerman theory of symplectic structure we require that the Jacobi equation

$$(\partial \bar{\partial} u)^{n-1} \partial \bar{\partial} \delta u = 0 \quad (10)$$

must be satisfied in addition to  $CMA_n$  itself. Then the Witten-Zuckerman symplectic 2-form is obtained by applying the exterior-functional derivative to  $\alpha$ . We find that the symplectic 2-form  $\omega = \delta\alpha$  is given by

$$\begin{aligned} \omega = & \frac{1}{2n!} \delta u (\partial \bar{\partial} u)^{n-1} \bar{\partial} \delta u \\ & + \frac{n-1}{2n!} \delta u \bar{\partial} u (\partial \bar{\partial} u)^{n-2} \partial \bar{\partial} \delta u \\ & - \frac{n-1}{2(n+1)!} \bar{\partial} \delta u \bar{\partial} u (\partial \bar{\partial} u)^{n-2} \partial \delta u \\ & + \frac{n-1}{2(n+1)!} \bar{\partial} \delta u \partial u (\partial \bar{\partial} u)^{n-2} \bar{\partial} \delta u \\ & + \frac{(n-1)(n-2)}{2(n+1)!} \bar{\partial} \delta u \partial u \bar{\partial} u (\partial \bar{\partial} u)^{n-3} \partial \bar{\partial} \delta u. \end{aligned} \quad (11)$$

It can be directly verified that  $\omega$  satisfies the closed

$$\delta \omega = 0 \quad (12)$$

and conserved

$$\partial \omega + (-1)^n \bar{\partial} \bar{\omega} = 0 \quad (13)$$

properties of the Witten-Zuckerman symplectic structure on  $\Lambda^2(\mathcal{F}(\mathcal{M})) \otimes [\Lambda^{n-1,n}(\mathcal{M}) \oplus \Lambda^{n,n-1}(\mathcal{M})]$ . The symmetry group of  $CMA_n$  is the group of holomorphic changes of the independent variables with unit Jacobian [8]. As in the case of its nearest relative, namely the group of diffeomorphisms, this is an infinite group. The symplectic 2-form (11) should be expressible as a Lie-Poisson structure associated with this group. That is, it must come from the co-adjoint action of vector fields belonging to the Lie algebra of the group of holomorphic changes of the independent variables with unit Jacobian.

### 3 Hilbert's variational principle

Kähler metrics with unit determinant, a requirement identical to  $CMA_n$ , are Ricci-flat. Since this condition is the same as the Euclideanized Einstein field equations for vacuum we are led to a second variational principle for  $CMA_n$  which is simply Hilbert's Lagrangian [15]. First we recall that the Kähler metric is given by

$$g_{i\bar{k}} = u_{i\bar{k}} \quad (14)$$

through the definition (3) and we note that

$$\mu = \sqrt{g} \quad (15)$$

must be nonzero. Therefore we must exclude the homogeneous complex Monge-Ampère equation from this part of the discussion. Then it is a standard result in differential geometry [16] that the Ricci tensor for Kähler metrics is given by

$$R_{i\bar{k}} = (\ln \mu)_{i\bar{k}} \quad (16)$$

which makes manifest the important role  $CMA_n$  plays in Kähler geometry.

The Hilbert Lagrangian density is

$$\mathcal{L}_H = \sqrt{g} R \quad (17)$$

where  $R = g^{i\bar{k}} R_{i\bar{k}}$  is the scalar of curvature formed out of the Riemann tensor. From eq.(16) and the definition of the contravariant metric, it can be verified that for Kähler metrics the Lagrangian (17) can be written as the  $(n, n)$ -form

$$L_H = \frac{1}{n!} (\partial \bar{\partial} u)^{n-1} \partial \bar{\partial} \ln \mu \quad (18)$$

but this is a divergence. Another way of seeing this divergence property, which has not been generally remarked on, is through eq.(16) and the following remarkable identity:

*Lemma*

For Kähler metrics

$$(\sqrt{g}g^{i\bar{k}})_{\bar{k}} = 0 \quad (19)$$

is an identity. Proof is by direct calculation which is immediate through the observation that  $\sqrt{g}g^{i\bar{k}}$  is given by the cofactors of  $u_{i\bar{k}}$ .

Hilbert's Lagrangian  $(n, n)$ -form (18) can be written as

$$L_H = \partial \bar{\partial} Z \quad (20)$$

in several different ways

$$Z = \frac{1}{n!} \begin{cases} u (\partial \bar{\partial} u)^{n-2} \partial \bar{\partial} \ln \mu, \\ \ln \mu (\partial \bar{\partial} u)^{n-1}, \\ \bar{\partial} u (\partial \bar{\partial} u)^{n-1} \partial \ln \mu, \end{cases} \quad (21)$$

which are alternative statements of the fact that for Kähler metrics the Hilbert Lagrangian is a topological invariant. Its first variation vanishes identically and we are left with only the boundary terms. Even though  $CMA_n$  does not emerge as the Euler-Lagrange equations from (18) an examination of the divergence terms is interesting because it is precisely these boundary terms in the first variation of the action that are important in the Witten-Zuckerman construction of the symplectic 2-form. For this purpose we write  $L_H$  in the form

$$L_H = \partial X + (-1)^n \bar{\partial} \bar{X} \quad (22)$$

$$X = \frac{1}{2n!} \bar{\partial} u (\partial \bar{\partial} u)^{n-2} \partial \bar{\partial} \ln \mu, \quad (23)$$

skipping other alternatives manifest in eq.(21) because they will yield degenerate results for symplectic structure. From the boundary terms using  $\delta \mu = 0$  by eqs.(10) and (15), we find that

$$\omega_2 = \bar{\partial} \delta u \partial \bar{\partial} \delta u (\partial \bar{\partial} u)^{n-3} \partial \bar{\partial} \ln \mu, \quad n > 2 \quad (24)$$

is the second Witten-Zuckerman symplectic 2-form on  $\Lambda^2(\mathcal{F}(\mathcal{M})) \otimes [\Lambda^{n-1,n}(\mathcal{M}) \oplus \Lambda^{n,n-1}(\mathcal{M})]$ . Physically the most interesting case of complex Monge-Ampère equation is the case  $n = 2$  but then  $\omega_2$  vanishes identically.

We have arrived at the symplectic 2-form (24) in an unconventional way. However, the ultimate justification for any result is a direct check of its properties. Eq.(24) satisfies all the properties required of a symplectic 2-form for  $CMA_{n>2}$  excluding the homogeneous case. Namely the check of the closed and conserved property of  $\omega_2$  given by eqs.(12), (13) is immediate. It is remarkable that for  $n = 2$ , the physically interesting case,  $\omega_2$  vanishes identically. For  $n > 2$  the symplectic 2-form vanishes only on shell. This property defines a Lagrangian submanifold [17], [18]. The symplectic 2-form expressed in terms of action angle variables vanishes on shell for integrable dynamical systems and the phase space is reduced to a Lagrangian submanifold. In eq.(24) we have the infinite dimensional analogue of this situation.

## 4 Complex Monge-Ampère-Liouville equation

The results we have presented above can be immediately extended to the complex Monge-Ampère-Liouville equation,  $\mu = \kappa e^{\Lambda u}$ , which will henceforth be referred to as  $CMAL_n$

$$\frac{1}{n!}(\partial\bar{\partial}u)^n = \kappa e^{\Lambda u} * 1 \quad (25)$$

that governs Einstein-Kähler metrics satisfying  $R_{i\bar{k}} = \Lambda g_{i\bar{k}}$ . The Lagrangian (6) is now modified to the form

$$L = \frac{1}{(n+1)!} \partial u \bar{\partial} u (\partial\bar{\partial}u)^{n-1} + \frac{\kappa}{\Lambda} (e^{\Lambda u} - 1) * 1 \quad (26)$$

and it can be verified that  $\omega$  in eq.(11) remains unchanged as the first Witten-Zuckerman symplectic 2-form for  $CMAL_n$ .

The analysis of the second Witten-Zuckerman symplectic 2-form for  $CMAL_n$  starts with the appropriate modification of the Hilbert Lagrangian

$$L_\Lambda = \frac{1}{n!} \left[ (\partial\bar{\partial}u)^{n-1} \partial\bar{\partial} \ln \mu - \Lambda (\partial\bar{\partial}u)^n \right] \quad (27)$$

and once again we find that it is a total divergence. As in the case of eq.(21) it can be written as a divergence in as many different ways. The result for the second Witten-Zuckerman symplectic 2-form is given by

$$\omega_{2\Lambda} = \bar{\partial}\delta u \partial\bar{\partial}\delta u (\partial\bar{\partial}u)^{n-3} \partial\bar{\partial} \ln \mu - \Lambda \bar{\partial}\delta u \partial\bar{\partial}\delta u (\partial\bar{\partial}u)^{n-2} \quad (28)$$

again with the proviso  $n > 2$ . Just as in the case of  $\omega_2$  we find that  $\omega_{2\Lambda}$  also vanishes when the Einstein-Kähler condition is satisfied. This is manifest when we put  $\mu = \kappa e^{\Lambda u}$  in eq.(28). Hence, as in the case of  $CMA_n$ , the second Witten-Zuckerman symplectic 2-form (28) for  $CMA L_n$  also defines a Lagrangian submanifold.

## 5 Conclusion

We have considered the covariant symplectic structure of complex Monge-Ampère and Monge-Ampère-Liouville equations in the covariant framework of the Witten-Zuckerman formalism adapted to holomorphic and anti-holomorphic differential forms. The Lagrangians (6) and (26) directly lead to the non-degenerate Witten-Zuckerman symplectic 2-form (11) in arbitrary dimension and for all cases elliptic, hyperbolic and homogeneous.

In order to prove the complete integrability of  $CMA_n$  we need two such symplectic 2-forms and use the theorem of Magri [19]. To this end we considered the Hilbert action principle (18) and (27) for Euclideanized vacuum Einstein field equations which are satisfied by virtue of  $CMA(L)_n$  and arrived at the symplectic 2-forms (24) and (28). Since Hilbert's Lagrangian is a divergence for Kähler metrics, it serves as a topological invariant rather than a Lagrangian for the Euclideanized Einstein field equations. Nevertheless, we were able to obtain the symplectic 2-forms (24) and (28) because only the boundary terms in the first variation of the action play a significant role in the Witten-Zuckerman construction. For  $n > 2$  they satisfy all the properties required of a symplectic 2-form for the complex elliptic, or hyperbolic Monge-Ampère-(Liouville) equation:  $CMA(L)_{n>2}$  admits bisymplectic structure.

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